Tidal Wave Propagation in a Rotating Conducting Fluid with a Magnetic Field

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1. Introduction

IN discussing the propagation of tidal waves in a nonrotating system, Lamb[1] has shown that viscosity damps waves that are capable of propagating and inhibits the propagation of such waves if the viscosity is large enough. This is because of the dissipation of energy due to viscosity. However, in a rotating system, viscosity may play a much different role. It is shown by Duty[2] that, in a rotating system, viscosity (if it is large enough) can interact with rotation and encourage tidal wave propagation instead of damping it. The physical reason for this lies in the fact that under certain circumstances viscosity plays the dual role of damping disturbances and at the same time extracting energy from the mean flow to feed into the disturbance, thus producing instability. This mechanism is discussed at length by Lin[3] in connection with boundary-layer instability.

On the other hand, Chandrasekhar,[4] while studying the thermal instability of a layer of a rotating electrically conducting fluid in the presence of a magnetic field, has shown that rotation can interact with the magnetic field to produce instability. Although in some respects the effects of rotation and a magnetic field acting separately are alike, viz., they both inhibit the onset of instability, they do not necessarily reinforce each other when acting together. On the contrary, they may sometimes oppose each other. Thus viscosity enhances the onset of instability if rotation is present, and it is also known that a magnetic field imparts to the fluid some characteristics of viscosity. Hence it follows that, even though the two acting separately inhibit the onset of instability, they might conspire to produce instability when acting jointly. Again rotation brings into play a component of vorticity along its direction, and, for large rotation, streamlines are closely wound spirals with motions mainly confined to planes perpendicular to the direction of rotation. But a magnetic field does not induce such a component of vorticity, and the tension in the lines of force tends to prevent motions transverse to them. This shows that the two fields do not produce comparable effects. From these considerations it is reasonable to think that, like thermal instability, a rotation interacting with a magnetic field might give rise to an enhanced tidal wave propagation. The present note reveals that for large Hartman numbers (as in the case of liquid mercury) rotation and magnetic field do give rise to such an instability.

2. Governing Equations

We assume that a plane horizontal sheet of an electrically conducting fluid (of magnetic permeability μ_e and electrical conductivity σ_0) of depth H is rotating about a vertical axis with an angular velocity Ω in the presence of a uniform vertical magnetic field H_0. Let us introduce a rectangular set of axes, rotating with the fluid, as the axis of z pointing vertically upward, and let us denote the components of fluid velocity $\mathbf{q}$ at a point by (u, v, w) along OX, OY, and OZ, respectively.

According to the customary tidal wave approximations, we shall ignore the substantial derivative Dw/Dt in the vertical momentum equation and also the effect of viscosity on the vertical component of motion which is in agreement with the neglected term $\partial w/\partial t$. It may be mentioned that the assumption of negligible vertical acceleration will be valid if $\Omega^2 r/g \gg 1$, where r is the greatest distance of any part of the sheet from the axis of rotation. The influence of viscosity is, however, retained in the horizontal momentum equations. In addition, we take the fluid as a liquid metal (viz., liquid mercury or sodium) for which the magnetic diffusivity $\eta = 1/(4\pi\mu_e\sigma_0)$ is very large (because σ_0 is very small), e.g., $\eta \sim 7.5 \times 10^3$ cm^2/sec for liquid mercury. Since the velocities are also small, the magnetic Reynolds number will indeed be very much less than 1. Under these conditions, the magnetic field lines slip freely through the fluid and convection is unable to make the magnetic field deviate appreciably from the imposed field H_0, so that the induced magnetic field is negligible compared with the applied field. The magnetic force $\mathbf{F}$ in the absence of excess charges is then given by

$$\mathbf{F} = \mu_e \mathbf{j} \times H_0 i_z \tag{1}$$

where i_z is the unit vector along the z axis and $\mathbf{j}$ is the current density. If the external electric field is zero and the induced electric field is negligible (which is justified since the effect of polarization of the conducting liquid is in general very small), the current density is related to the velocity by Ohm's law as follows:

$$\mathbf{j} = \sigma_0 \mu_e \mathbf{q} \times H_0 i_z \tag{2}$$

Combining (1) and (2), the force components f_x and f_y are found to be

$$F_x = -\sigma_0 B_0^2 u \qquad F_y = -\sigma_0 B_0^2 v \tag{3}$$

where $B_0 = \mu_e H_0$. With the foregoing approximations, the horizontal equations of momentum are

$$\frac{\partial u}{\partial t} - 2\Omega v - \Omega^2 x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma_0 B_0^2 u}{\rho} \tag{4}$$

$$\frac{\partial v}{\partial t} + 2\Omega u - \Omega^2 y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\sigma_0 B_0^2 v}{\rho} \tag{5}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{6}$$

The medium outside the fluid is assumed to be nonconducting. Thus, when there is relative equilibrium under gravity, pressure p_0 is given by

$$p_0 = (\rho\Omega^2/2)(x^2 + y^2) - \rho g z + \text{const} \tag{7}$$

so that on the free surface

$$z_0 = (\Omega^2/2g)(x^2 + y^2) + \text{const} \tag{8}$$

If the disturbed free surface is taken as $z = \zeta + z_0$ then, since the vertical acceleration is assumed to be small compared with g, the pressure at any point is given by

$$p = \rho g(\zeta + z_0 - z) \tag{9}$$

Combining (8) and (9), we can write (4) and (5) as

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \zeta}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma_0 B_0^2 u}{\rho} \tag{10}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \zeta}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\sigma_0 B_0^2 v}{\rho} \tag{11}$$

where $f(= 2\Omega)$ is the Coriolis parameter. The equation of continuity is

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0 \tag{12}$$

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In order to seek tidal wave solutions of Eqs. (10–12), we write the time dependence of u, v, and ζ in the form

$$u = e^{i\sigma t}\,\bar{u}(x, y) \qquad v = e^{i\sigma t}\,\bar{v}(x, y) \qquad \zeta = e^{i\sigma t}\,\bar{\zeta}(x, y) \tag{13}$$

This gives from (10) the expression for $\bar{v}(x, y)$

$$\bar{v} = f^{-1}\cdot\left[\left(i\sigma + \frac{\sigma_0 B_0^2}{\rho}\right)\bar{u} + g\frac{\partial\bar{\zeta}}{\partial x} - \nu\nabla^2\bar{u}\right] \tag{14}$$

We shall further restrict ourselves to waves propagating along x axis in a fluid of constant depth $H(x, y) = h$. It should be noted that the depth will not be uniform unless the bottom of the channel follows the curvature of the surface given by (8). Consequently, this amounts to the consideration of a paraboloidal bottom. Hence, the solutions (13) assume the form

$$u = Ae^{i(\sigma t - Kx)} \qquad v = Be^{i(\sigma t - Kx)} \qquad \zeta = Ce^{i(\sigma t - Kx)} \tag{15}$$

These give, from (12),

$$C = hKA/\sigma \tag{16}$$

whereas Eqs. (10) and (11) yield

$$Ai\sigma - fB - igCK = -A\nu[K^2 + (\sigma_0 B_0^2/\rho\nu)] \tag{17}$$

$$Bi\sigma + fA = -B\nu[K^2 + (\sigma_0 B_0^2/\rho\nu)] \tag{18}$$

Elimination of A, B, and C from (16, 17, and 18) gives

$$\sigma^3 - 2i\left(\nu K^2 + \frac{\sigma_0 B_0^2}{\rho}\right)\sigma^2 - \left[\left(\nu K^2 + \frac{\sigma_0 B_0^2}{\rho}\right)^2 + K^2 gh + f^2\right]\sigma + iK^2 gh\left(\nu K^2 + \frac{\sigma_0 B_0^2}{\rho}\right) = 0 \tag{19}$$

as the required dispersion relation for the propagation of the tidal waves.

3. Discussion of the Dispersion Relation

We shall first study the case when the coupling between rotation and the magnetic field is dominant. Introducing the following nondimensional quantities

$$\sigma^* = \frac{\sigma h^2}{\nu} \qquad K^* = Kh \qquad M = \frac{\sigma_0 B_0^2 h^2}{\rho\nu}$$
$$N = \frac{gh^3}{\nu^2} \qquad f^* = f\left(\frac{h}{g}\right)^{1/2} \tag{20}$$

Eq. (19) becomes

$$\sigma^{*3} - 2i(K^{*2} + M)\sigma^{*2} - [(K^{*2} + M)^2 + N(K^{*2} + f^{*2})]\sigma + iNK^{*2}(K^{*2} + M) = 0 \tag{21}$$

According to the definition in (20), M is the square of the Hartman number $B_0 h(\sigma_0/\rho\nu)^{1/2}$. To reduce this to an equation having all real coefficients, we put $\sigma^* = -i\gamma$ so that (21) becomes

$$\gamma^3 + a\gamma^2 + b\gamma + c = 0 \tag{22}$$

where

$$a = 2(K^{*2} + M) \qquad b = (K^{*2} + M)^2 + N(K^{*2} + f^{*2})$$
$$c = NK^{*2}(K^{*2} + M) \tag{23}$$

The nature of the roots of (22) can be seen[5] from the discriminant Δ of the equation where

$$\Delta = 4p^3 + 27q^2 \qquad p = b - \frac{a^2}{3}$$
$$p = c - \frac{ba}{3} + \frac{2a^3}{27} \tag{24}$$

Using (23) and (24), the discriminant is given by

$$\Delta = 4N^2(K^2 + f^2)^3 + N^2(K^2 + M)^2 \times [8f^4 - 20K^2f^2 - K^4] + 4Nf^2(K^2 + M)^4 \tag{25}$$

where the asterisks are dropped for convenience.

Three cases are of interest: 1) $\Delta > 0$, i.e., one root is real but the other two roots are complex conjugates; 2) $\Delta = 0$, i.e., all the roots are real and two of them are equal; 3) $\Delta < 0$, i.e., all the roots are real and unequal. If $\Delta \leq 0$, as in the second and the third cases, the effect of the magnetic field is simply to damp the waves as they propagate along the x direction. However, if M (i.e., Hartman number) is very large, the discriminant in (25) can be approximately written as

$$\Delta \simeq 4Nf^2M^4 \tag{26}$$

and this is clearly positive. This shows, curiously enough, that, when rotation is present ($f \neq 0$), a magnetic field can interact with the rotation to give rise to an enhanced tidal wave propagation.

We next consider the case when the coupling between viscosity and rotation is dominant. Introducing the nondimensional variables

$$\sigma_1 = g^{-1/2}h^{1/2}\sigma \qquad f_1 = h^{1/2}g^{-1/2}f$$
$$N_1 = g^{-1/2}h^{-3/2}\nu \qquad K_1 = Kh \qquad M_1 = \frac{\sigma_0 B_0^2}{\rho}\left(\frac{h}{g}\right)^{1/2} \tag{27}$$

Eq. (19) becomes

$$\sigma_1^3 - 2i(N_1K_1^2 + M_1)\sigma_1^2 - [K_1^2 + f_1^2 + (N_1K_1^2 + M_1)^2]\sigma_1 + iK_1^2(N_1K_1^2 + M_1) = 0 \tag{28}$$

This reduces, upon putting $\sigma_1 = -i\gamma$ as before, to

$$\gamma^3 + 2(N_1K_1^2 + M_1)\gamma^2 + [K_1^2 + f_1^2 + (N_1K_1^2 + M_1)^2]\gamma + K_1^2(N_1K_1^2 + M_1) = 0 \tag{29}$$

The discriminant Δ of this cubic equation is given by

$$\Delta = 4(K_1^2 + f_1^2)^3 + 4f_1^2(N_1K_1^2 + M_1)^4 + (N_1K_1^2 + M_1)^2\cdot[8f_1^4 - 20K_1^2f_1^2 - K_1^4] \tag{30}$$

It is clear from the definitions in (27) that viscosity is characterised by the parameter N_1.

When viscosity is large (as in a treacle or pitch), i.e., when $N \gg 1$, $\Delta \simeq 4f_1^2N_1^4K_1^8$, which is positive. Thus tidal waves can propagate.

Let us consider the case of large viscosity and small rotation, i.e., $N_1^2f_1^2 \ll 1$ and $N_1 \gg 1$. In this case, $\Delta \sim -(N_1K_1^2 + M_1)^2\cdot K^4$, which is negative. Thus tidal waves can not propagate.

When both viscosity and Hartman number are small and we make the tidal wave approximation $K \ll 1$, then $\Delta = 4f_1^6 + 12K_1^2f_1^4 > 0$. Thus, in this situation, the tidal waves exhibit a behavior similar to that found for a slightly viscous fluid in the absence of rotation.

References

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